COMBIOATORICA 5 (2) (1985) 141—147

THE STRUCTION OF A GRAPH: APPLICATION TO *CN*-FREE GRAPHS

P. L. HAMMER, N. V. R. MAHADEV and D. DE WERRA*

Received 4 July 1983
Revised 15 October 1984

We consider the class of graphs characterized by the forbidden subgraphs C and N: C is the claw (unique graph with degree sequence (3, 1, 1, 1)) and N the net (unique graph with degree sequence (3, 3, 3, 1, 1, 1)). For this class of graphs (called CN-free) an algorithm is described for determining the stability number $\alpha(G)$. It is based on a construction associating with any CN-free graph G another CN-free graph G' such that $\alpha(G')=\alpha(G)-1$. Such a construction reducing the stability number is called a struction.

1. Introduction

The purpose of this paper is to show how for some classes of graphs G one can obtain with a polynomial algorithm the stability number $\alpha(G)$ by using a reduction technique. By this we mean a construction which associates with any graph G in some class F another graph G' in F with $\alpha(G')=\alpha(G)-1$. Such a construction reducing the stability number will be called a *struction.*

This approach is similar in spirit to the struction procedure described in [2] for general graphs; we will show here that the class of graphs defined by two forbidden subgraphs C and N is closed under an adequate struction. Here C will be the claw, i.e. the unique graph with degree sequence (3,1,1,1) (see Fig. 1a) and N will be the net, i.e. the unique graph with degree sequence (3,3,3,1,1,1) (see Fig.

a) the claw $C=(i;\, j, k, l)$ *b) the net* $N=(i, j, k;\, i', j', k')$

Fig. 1

* This work was completed while this author was visiting the Dept. of Combinatories and Optimization at the University of Waterloo, Ontario. The support of NSERC is gratefully acknowledged.

AMS subject classification (1980): 05 C 99, 68 E 10

1b). Loops and parallel edges will be excluded. Graphs having no induced C or N will be called CN-free. Such graphs have been studied with respect to hamiltonicity [1]. One should observe that the class of CN-free graphs includes an infinity of 2-connected graphs with $\alpha(G)\geqq 3$; one such family is obtained for instance by taking a chordless cycle C_k on nodes $a_1, a_2, \ldots, a_k$ and a chordless cycle C_k on nodes $b_1, b_2, \ldots, b_k$. One links each node a_i in C_k with nodes b_i, b_{i+1} in C_k (here indices are taken modulo k).

The focus of the paper lies essentially in the presentation of a new method for computing the stability number of graphs. Here this approach is adapted to a subclass of claw-free graphs. For general claw-free graphs, a polynomial algorithm has been given in [5, 6]; our approach for this subclass of claw-free graphs is quite different and might lead to an apparently new type of algorithm for various classes of graphs. A struction has been described previously for a subclass of CN-free graphs, the so-called CAN-free graphs [4]. We present in the next section another struction which is adapted to CN-free graphs. One of the differences with the struction for CAN-free graphs is that during the construction an adequate ordering of the nodes has to be defined.

In Section 3 we will show that the class of CN-free graphs is closed under the struction.

Unless otherwise specified we will use the graph terminology of [3].

2. Struction of CN-free graphs

Before describing the stability number reducing operation (i.e. the struction), we need to introduce a few notations. $[i, j]$ will indicate the presence of an edge between nodes i and j, while $\overline{[i, j]}$ will denote the absence of such an edge (or the presence of a "nonedge"). For a node a, $N(a)=\{j \mid [a, j]\}$ is the set of neighbours of a; $\overline{N(a)}=\{j \mid \overline{[a, j]}\}$, $N[a]=\{a\}\cup N(a)$.

An arbitrary fixed node 0 will be crucial in the struction; so we define $N_0[a]=N[a]\cap N(0)$ and $\overline{N_0(0)}=\overline{N(a)}\cap N(0)$. Let $\lesssim$ be a partial order defined on $N(0)$ by $a\lesssim b$ if $N_0[a]\subseteqq N_0(b]$. We shall write $a\approx b$ if $a\gtrsim b$ and $b\gtrsim a$. In the struction, the nodes in $N(0)$ will be numbered from 1 to $|N(0)|$ in an adequate way; it will be convenient to refer to these nodes simply with their associated number. So we will have a total order $<$ on the nodes in $N(0)$.

We now describe the struction for CN-free graphs.

(a) *Preliminaries.*

1) choose any node 0 in $G=(X, U)$; 0 will be the center of the struction
2) number the nodes in $N(0)$ from 1 to $|N(0)|$ in such a way that
   (i) if $a\lesssim b$ and $a\not\gtrsim b$ then $a<b$
   (ii) if $a\approx b, a\not\lessgtr x, x\not\lessgtr a$ and $a<b$, then either $x<a$ or $x>b$.
3) Let $I^*=\{i\in N(0) \mid \exists j\in N(0)$ with $j>i$ and $\overline{[i, j]}\}$.

(b) *Construction of G'.*

4) Introduce the subgraph R induced by $X-N[0]$
5) For each $i\in I^*$ introduce a new node i^*

6) link every pair of new nodes
7) link a new node i^* to node r in R if in G we have either $[i, r]$ or $[j, r]$ for every $j > i$ with $j \in \overline{N_0}(i)$.

Remark. One should observe that a numbering satisfying 2(i) and 2(ii) can always be found; it is simply a numbering which does not contradict the partial order $\leqq$; furthermore if 2 nodes a, b are equivalent with respect to $\leqq$ one should not give to a node x not comparable to a or b a number between a and b.

The construction is illustrated in Fig. 2 where a graph G and the resulting graph G' are shown.

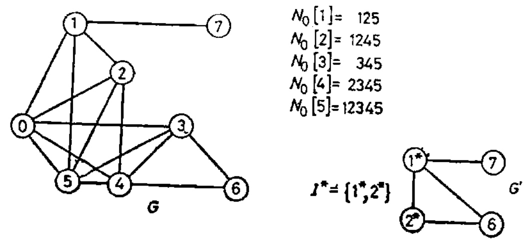

Fig. 2. The struction transforming G into G'

Proposition 2.1. *Let G be a CN-free graph with* $\alpha(G) > 1$, 0 *an arbitrary node of G and G′ a graph resulting from a struction, centered at* 0; *then* $\alpha(G') = \alpha(G) - 1$.

Proof. A) We shall first show that if $S \neq \emptyset$ is a stable set in G, there exists a stable set S' in G' with $|S'| = |S| - 1$. Observe that since G is claw-free then $|S \cap N[0]| \leqq 2$. If $S \cap N[0] = \emptyset$, take $S' = S - x$ where x is any node in $S \cap R$; if $S \cap N[0] = \{i\}$, then $S' = S - \{i\}$. Finally if $S \cap N[0] = \{i, j\}$, then we may assume $0 < i < j$ and $[\overline{i, j}]$ holds; we take $S' = (S - \{i, j\}) \cup \{i^*\}$. This set is stabel; assume we have $[i^*, r]$ for some r in $R \cap S$; r cannot be in $N(i)$ since $S \ni i$ was stable, so since r was linked to i^* in G' we had in G $[r, l]$ for all $l > i$ in $\overline{N_0(i)}$; in particular r was linked to j; this is also impossible because $S \ni j$ and S was stable in G. Hence we cannot have $[i^*, r]$ for $i^*, r \in S'$. So in every case S' satisfies $|S'| = |S| - 1$ and is stable.

B) Let us show now that for any stable set S' in G' we can find a stable set S in G with $|S| = |S'| + 1$. Let K^* be the clique of new nodes introduced in G'. Obviously $|S' \cap K^*| \leqq 1$. If $S' \cap K^* = \emptyset$, then $S' \subset R$ and we may take $S = S' \cup \{0\}$. If $S' \cap K^* = \{i^*\}$, then since i^* is a node of G', we have in G, $N^* = \{j \mid j > i, j \in \overline{N_0}(i)\} \neq \emptyset$. We shall show that there exists in N^*, a j for which $S = (S' - \{i^*\}) \cup \{i, j\}$ is stable in G.

(a) Now clearly, we will have $[\overline{i, r}]$ for any $r \in R \cap S$; this holds because $i^*, r \in S'$ and hence $[\overline{i^*, r}]$ in G' which means $r \notin N(i)$ in G.

(b) Now, for each j in N^* we have $[j, r]$ for at most one $r \in R \cap S$ (otherwise, if $[j, r], [j, r']$ for $r, r' \in R \cap S$ then $(j;\ 0, r, r')$ forms a claw).

(c) Assume now that for each j_k in N^* there is a node r_k in $R \cap S$ with $[j_k, r_k]$, this implies that $|N^*| \geqq 2$ (because if $N^* = \{j_1\}$ a stable set S' containing i^* cannot contain r_1 since $[i^*, r_1]$ in G'.

(d) Choose j_1 in N^*; there is an $r_1 \in R \cap S$ with $[j_1, r_1]$ by (c); since $\overline{[i^*, r_1]}$ in G', there is a j_2 in N^* with $\overline{[j_2, r_1]}$; by (c) there is r_2 in $R \cap S$ with $[j_2, r_2]$. We have $r_1 \neq r_2$ and so $[0, i]$, $[0, j_1]$, $[0, j_2]$, $\overline{[i, j_1]}$, $\overline{[i, j_2]}$, $[j_1, r_1]$, $[j_2, r_2]$, $\overline{[j_2, r_1]}$. Also $[j_1, j_2]$, otherwise $(0;\, i_1, j_1, j_2)$ is a claw.

From (b) and $[j_1, r_1]$ follows $\overline{[j, r_2]}$. Furthermore from (a) we have $\overline{[i, r_1]}$, $\overline{[i, r_2]}$. Also since $r_1, r_2 \in R \cap S$, $\overline{[r_1, r_2]}$. But now $(0, j_1, j_2, i, r_1, r_2)$ define an induced net of G and this is impossible. Hence there must exist a $j > i$ in $\overline{N}_0(i)$, with $\overline{[j, r]}$ for all $r \in R \cap S$. So G contains a stable set S of the form $(S' - (i^*)) \cup \{i, j\}$.
C) Since A) implies $\alpha(G') \geqq \alpha(G) - 1$ and B) implies $\alpha(G) \geqq \alpha(G') + 1$, we have $\alpha(G') = \alpha(G) - 1$. ∎

Remark. In the above proof, one does not use properties (i) and (ii) of the ordering of the nodes in $N(0)$.

Proposition 2.2. *If G is a nontrivial CN-free graph, the struction gives a graph G' with at least 2 nodes less than G.*

Proof. It follows immediately from the description of the struction that each node i in $N(0)$ will give at most one new node i^* in G', since 0 will not be in G' and the last node j in $N(0)$ will not give any j^*, the result follows. ∎

From Proposition 2.1 and 2.2, one deduces that by repeatedly applying the struction to a CN-free graph, we get a simple polynomial algorithm giving the stability number of G, provided the class of CN-free graphs is closed under the struction. This will be established in the next section.

3. Closedness of the family of CN-free graphs

In this section we shall prove that the struction described in section 2 transforms a CN-free graph into another CN-free graph. If a is a node in $N(0)$, b is in $\overline{N}_0(a)$ and $b > a$, then we shall say that b is a follower of a.

Unless otherwise specified we shall assume throughout this section that
(i) G is a CN-free graph
(ii) G' is a transform of G (i.e. the result of a struction centered at 0)
(iii) if $a, b, c, \ldots$ denote new nodes, then $a', b', c', \ldots$ are nodes in G corresponding to $a, b, c, \ldots$ respectively and $a'', b'', c'', \ldots$ are followers of $a', b', c', \ldots$ respectively
(iv) $r_1, r_2, \ldots$ are nodes in R.

Lemma 3.1. *$[a, r_1]$, $[a, r_2]$ and $\overline{[r_1, r_2]}$ in G' implies $[a', x]$, $[a'', y]$, $\overline{[a', y]}$ and $\overline{[a'', x]}$ in G where $\{x, y\} = \{r_1, r_2\}$ and a'' is any follower of a'.*

Proof. For $i = 1, 2$ $[a, r_i] \Rightarrow [a', r_i]$ or $[a'', r_i]$. But $[a', r_1]$ and $[a', r_2]$ is impossible since otherwise $(a';\, 0, r_1, r_2)$ is a claw in G. Similarly $[a'', r_1]$ and $[a'', r_2]$ is impossible.

Lemma 3.2. *$\overline{[a, r_1]}$, $\overline{[a, r_2]}$ and $\overline{[r_1, r_2]}$ in G' implies $\overline{[a', r_1]}$, $\overline{[a', r_2]}$, $\overline{[a'', r_1]}$, $\overline{[a'', r_2]}$ in G for some follower a'' of a'.*

Proof. By definition of the struction, $[\overline{a', r_1}]$, $[\overline{a', r_2}]$, $[\overline{a_1, r_1}]$ for a follower a_1 of a' and $[\overline{a_2, r_2}]$ for a follower a_2 of a'. If $a_1 = a_2$, the lemma is proved. So assume $a_1 \neq a_2$ and $[a_1, r_2]$, $[a_2, r_1]$. But then, since $[a_1, a_2]$ in G, (otherwise $(0;\ a', a_1, a)$ is a claw), $(0, a_1, a_2;\ a', r_2, r_1)$ is a net. This case is impossible. ▮

The graph $B = (r, a, b, 0, y, x, s)$ will play an important role in the remaining of this section; it is represented in Fig. 3. We recall that we have defined a partial order $\leqq$ on $N(0)$ by $a \leqq b$ if $N_0[a] \subseteq N_0[b]$.

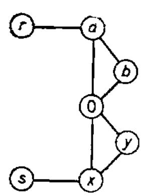

Fig. 3. The graph $B = (r, a, b, 0, y, x, s)$

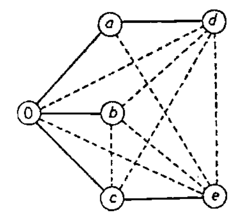

Fig. 4. The configuration $H(0, a, b, c, d, e)$ (heavy lines represent edges and dotted lines nonedges)

Lemma 3.3. *If $B = (r, a, b, 0, y, x, s)$ is an induced subgraph of G, then $a \leqq b$ and $x \leqq y$.*

Proof. We show $a \leqq b$; a similar reasoning shows that $x \leqq y$. Let $c \in N_0(a) - N_0(b)$, then $[c, r]$ (otherwise $(a;\ b, c, r)$ is a claw), $[c, y]$ (otherwise $(0;\ b, c, y)$ is a claw), $[c, x]$ (otherwise $(0;\ b, c, x)$ is a claw), hence $[c, s]$ (otherwise $(0, c, x;\ s, r)$ is a net). But then $(c;\ 0, r, s)$ is a claw. So this case is impossible, hence $N_0(a) - N_0(b) = \emptyset$, i.e. $N_0(a) \subseteq N_0(b)$ and $a \leqq b$. ▮

Lemma 3.4. *If G contains the configuration $H(0, a, b, c, d, e)$ of Fig. 4, then $[\overline{a, c}]$.*

Proof. If $[a, c]$, then $(0, a, c;\ b, d, e)$ is a net unless $[a, b]$; however $[a, c]$, $[a, b]$ imply that $(a;\ b, c, d)$ is a claw. Hence $[\overline{a, c}]$. ▮

We can now state:

Proposition 3.1. *If G' is a transform of a CN-free graph G, then G' is claw-free.*

Proof. Assume that G' contains a claw $(a;\ b, c, d)$. At least one of a, b, c, d is a new node. By construction the new nodes form a clique. We have 3 cases to consider

Case 1: b alone is a new node.

Let b' correspond to b in G. By lemma 3.2, there exist a follower b'' of b' such that $[\overline{b'', c}]$, $[\overline{b'', d}]$, $[\overline{b', c}]$ and $[\overline{b', d}]$. Now $[a, b]$ in G' implies $[a, b']$ or $[a, b'']$ in G'. In either case there is a claw in a. This case is not possible.

Case 2: a alone is a new node.

Let a' correspond to a. By lemma 3.1, $[a', b]$, $\overline{[a', c]}$, $\overline{[a'', b]}$, $[a'', c]$, also $[a', d]$ or $[a'', d]$; but $[a', d]$ creates claw $(a';\ 0, b, d)$ and $[a'', d]$ creates claw $(a'';\ 0, c, d)$. This case is impossible.

Case 3. a and b are the only new nodes.

By lemma 3.1, $[a', c]$, $\overline{[a', d]}$, $\overline{[a'', c]}$ and $[a'', d]$. By lemma 3.2 $\overline{[b', c]}$, $\overline{[b', d]}$, $\overline{[b'', c]}$ and $\overline{[b'', d]}$.

Subcase 3.1. $[a', b']$.

Then we have $\overline{[a', b'']}$, otherwise (a', b', b'', c) is a claw, then $[a'', b'']$ and $\overline{[a'', b']}$ for similar reasons. Then $(d, a'', b'', 0, b', a', c)$ is an induced B. By lemma 3.3, $a' \leq b'$, hence $b' < b''$ implies $a' < b''$, so b'' is a follower of a'; this implies $[b'', d]$ (because we had $[a, d]$ in G). This contradicts $\overline{[b'', d]}$.

Subcase 3.2. $\overline{[a', b']}$

Since G is claw-free, $[a', b'']$, $[a'', b']$ and hence $\overline{[a'', b'']}$ (otherwise there is a claw $(a'';\ b', b'', d)$). Then $(c, a', b'', 0, b', a'', d)$ is an induced B; hence $a'' \leq b'$. So $a' < a''$ implies $a' < b'$, so b' is a follower of a' and we must have $[b', d]$ (since we had $[a, d]$ in G). This contradicts $\overline{[b', d]}$.

So case 3 is not possible; since all remaining cases are symmetric to cases 1, 2, 3, G' cannot have an induced claw containing at least one new node. Hence G' is claw-free. ∎

It only remains to show that G' is also net-free.

Proposition 3.2. *If G and G′ are defined as in theorem 3.1, then G′ is net-free.*

Sketch of proof. We shall only indicate how proposition 3.2 can be proved by enumerating a collection of cases; the details will be left to the reader. Assume that $(b, c, d;\ a, e, f)$ is an induced net N in G'; at least one node of N is a new node. We have 4 cases to consider

Case 1: a alone is a new node.

Let a' correspond to a; we have $\overline{[a', x]}$ where $x = c, d, e, f$. By lemma 3.2, there is an a'' such that $\overline{[a'', c]}$ and $\overline{[a'', f]}$. Now $\overline{[a', b]}$ otherwise $(b, c, d;\ a', e, f)$ is a net. Since $[a, b]$ in G, we have $[a'', b]$. But then $[a'', d]$ or $[a'', e]$ (otherwise $(b, c, d;\ a'', e, f)$ is a net). If $[a'', d]$, then $(d;\ a'', c, f)$ is a claw and if $[a'', e]$, then $(a'';\ 0, b, e)$ is a claw. So this case is impossible.

Case 2: among b, c, d exactly one node (say b) is new.

Using lemma 3.2 one shows that this case is impossible

Case 3: Among b, c, d exactly two nodes (say b and c) are new.

This case is also impossible as can be seen by applying lemmas 3.1, 3.2 and 3.4.

Case 4: the only new nodes are b, c, d.

Let b', c', d' be the nodes of G corresponding to b, c, d from lemma 3.2, $\overline{[b, e,]}$, $\overline{[b, f]}$, $[e, f]$ in G' imply $\overline{[b', e]}$, $\overline{[b'', e]}$, $\overline{[b', f]}$, $\overline{[b'', f]}$ for some follower b'' of b'. Similarly we have $\overline{[c', a]}$, $\overline{[c', f]}$, $\overline{[c'', a]}$, $\overline{[c'', f]}$ for some follower c'' of c' and also $\overline{[d', a]}$, $\overline{[d'', a]}$, $\overline{[d', e]}$, $\overline{[d'', e]}$ for some follower d'' of d'.

We have four subcases to consider

Subcase 4.1. $\overline{[b', a]}$, $\overline{[c', e]}$, *and* $\overline{[d', f]}$.

Subcase 4.2. $[b', a]$, $\overline{[c', e]}$, $\overline{[d', f]}$.

Subcase 4.3. $[b', a]$, $[c', e]$ *and* $\overline{[d', f]}$.

Subcase 4.4. $[b', a, [c', e]$ *and* $[d', f]$.

It is a routine task to verify that all these cases are impossible by using lemma 3.4.

There is no other case since by definition of the struction all new nodes form a clique.

Hence G' cannot contain an induced net. ∎

4. Final remarks

Although the proofs given in section 3 are rather long, the struction itself is a simple operation to perform on a CN-graph in particular. It is not difficult to see that this special version of the struction leads to an $0(n^3)$ algorithm. The general struction defined for any graph in [2] is also simple but does not lead as here to a polynomial algorithm since G' may in the worst cases have many more nodes than G. One should also mention that apparently there is no simple extension to the weighted case of this specialized struction.

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P. L. Hammer,

Rutgers University
RUTCOR, New Brunswick
NY 08903, U.S.A.

N. V. R. Mahadev

Dept. Mathematics
University of Winnipeg
Winnipeg, Manitoba, R3B 2E9
Canada

D. de Werra

Département de Mathématiques
Ecole Polytechnique Fédérale de
Laussanne
CH—1015 Laussanne
Switzerland